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NUMERICAL METHODS FOR DETERMINING AN UNKNOWN SPACEWISE-DEPENDENT SOURCE IN THE HEAT EQUATION

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ABSTRACT

This paper describes and compares two effective methods for the numerical solution of an unknown space dependent source function in the heat equation. One of these methods is obtained by differentiating the equation with respect to the time variable and using the finite-difference method, while the other is based on the trace type functional formulation of the problem.

Key Words: Unknown coeficients; Finite-difference method; Inverse Problems; Trace-Type Functional Formulation.

INTRODUCTION

We consider the following inverse problem of finding unknown source function $f(x)$ in the parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(x), \quad 0 < x < l, \quad 0 < t \le T, \quad (1)$$

subject to the boundary conditions

$$u(0,t) = h_0(t) \qquad u(l,t) = h_l(t), \qquad 0 \le t \le T, \quad (2)$$

initial condition

$$u(x,0) = u_0(x), \qquad 0 \le x \le l, \qquad (3)$$

and the overspecified condition

$$u(x,T) = \varphi(x), \qquad 0 \le x \le l. \qquad (4)$$

In the context of heat conduction or diffusion, where u represents the temperature or the concentration, the unknown function $f(x)$ is interpreted as a heat or material source, respectively. One of the important application are of the problem is pollution control, where the pollution source intensity $f(x)$ needs to be determined [1,2]. The problem (1)-(4) of determining an unknown heat source $f(x)$ in the heat conduction equation has been considered in many theoretical papers, notably [3-6], where the existence and uniqueness of a smooth solution of the inverse problem was studied. There are two types of numerical methods for solving this problem. One approach referred to in the literature as the method of output least squares is to assume that the unknown coefficient is of a specific functional form depending on some parameters and then, seek to determine the optimal parameter values so as to minimize an error functional based on the additional condition. The appeal of output least squares approach lies in the well-developed theory for dealing with optimization problems. On the other hand, it is usually not evident that the solution of the optimization problem solves the original problem and the error functional may be based on data which do not uniquely determine the unknown coefficient. The approaches considered in this paper are not of this type. The first method is to reformulate the equation (1) by differentiating it with respect to the time variable and using the finite-difference method for the numerical solution. In another method that we call as a Trace-Type Functional formulation the unknown coefficient is eliminated from the parabolic equation and the iterative Fixed Point Projection method is applied to solve the resulting problem.

FINITE DIFFERENCE WITH CONJUGATE-GRADIENT (METHOD I)

By differentiating equation (1) with respect to we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial^2 u}{\partial x \partial t}\right), \quad 0 < x < l, \quad 0 < t < T. \quad (5)$$

Then the reformulated problem consists of the equation (5) along with (2) and (4).

After determining $u(x,t)$ from this reformulated problem, the unknown source function f can be defined as

$$f(x) = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right). \qquad (6)$$

Let $\tau = \Delta t > 0$ and $h = \Delta x > 0$ be step lengths on time and space coordinates, $\overline{\omega}_\tau = \omega_\tau \cup \{T\} = \{t_n = n\tau, n = 0,1,\dots,N_0, \tau N = T\}$ and $\overline{\omega} = \{x | x = x_i = ih, i = 0,1,\dots,N, Nh = l\}$ denote the uniform partitions of [0, T] and [0, 1], respectively. The implicit finite difference approximation of the system of equations (5,\2,\3) can be written in the form

$$\frac{u^{n+1}-2u^{n}+u^{n-1}}{\tau^{2}}+A\frac{u^{n+1}-u^{n-1}}{2\tau}=0,\quad x\in\ \omega,\quad (7)$$

where $n=1,2,\dots,N-1$, ω represents the internal points of $\overline{\omega}$, $Au^{n}=-(au_{\bar{x}}^{n})_{x}=-\frac{1}{h}\left(k_{i+1/2}\frac{u_{i+1}^{n}-u_{i}^{n}}{h}-k_{i-1/2}\frac{u_{i}^{n}-u_{i-1}^{n}}{h}\right)$ is a fnite difference approximation of term $-\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)$ in (1) $a(x)=k(x-0.5h)$, $k_{i-1/2}$ and u_{i}^{n} are the approximations to $k(x-0.5h)$ and $u(x_i,t_n)$, respectively. Approximations of boundary and initial condition have the form

$$u_{0}^{n}=h_{0}(t_{n}),\qquad u_{N}^{n}=h_{l}(t_{n}),\qquad n=0,1,\dots,N,\qquad (8)$$

$$u_{i}^{0}=u^{0}(x_{i}),\qquad i=0,1,\dots,N,\qquad (9)$$

$$u_{i}^{N_{0}}=\varphi(x_{i}),\qquad i=0,1,\dots,N,\qquad (10)$$

The unknown f_i is defined from the approximation of (6):

$$f_{i}=\frac{u_{i}^{n+1}-u_{i}^{n}}{\tau}-\frac{1}{h}\left(k_{i+\frac{1}{2}}\frac{u_{i+1}^{n}-u_{i}^{n}}{h}-k_{i-\frac{1}{2}}\frac{u_{i}^{n}-u_{i-1}^{n}}{h}\right)\qquad (11)$$

for some $n=n_0$.

To solve the system of equations (7) - (10) the conjugate gradient method with Gauss symmetrization is applied. The details of this method can be found in [7].

TRACE-TYPE FUNCTION FORMULATION (METHOD II)

If the functions pair $(u,f(x))$ solves the inverse problem (1) - (4), then

$$\left.\frac{\partial u}{\partial t}\right|_{t=T}=\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)\Big|_{t=T}+f(x),\qquad 0<x<l,\qquad (12)$$

from which

$$f(x)=\left.\frac{\partial u}{\partial t}\right|_{t=T}-\frac{\partial}{\partial x}\left(k(x)\frac{\partial \varphi}{\partial x}\right)\Big|_{t=T}.\qquad (13)$$

Substituting (13) in (1) leads

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)+\left.\frac{\partial u}{\partial t}\right|_{t=T}-\frac{\partial}{\partial x}\left(k(x)\frac{\partial \varphi}{\partial x}\right)\Big|_{t=T},\ (14)$$

Which has to be solved subject to (2) and (3). Such representation is called the TTF(Trace Type Functional) formulation of problem (1) - (4). From the solution of this system the approximation solution of $f(x)$ can be determined from (12).

The implicit finite difference approximation of eq.(14) can be as follows;

$$\frac{u_{i}^{n+1}-u_{i}^{n}}{\tau}=\frac{1}{h}\left(k_{i+1/2}\frac{u_{i+1}^{n+1}-u_{i}^{n}}{h}-k_{i-1/2}\frac{u_{i}^{n+1}-u_{i-1}^{n+1}}{h}\right)+\frac{u_{i}^{M}-u_{i}^{M-1}}{\tau}-\frac{1}{h}\left(k_{i+1/2}\frac{\varphi_{i+1}^{M}-\varphi_{i}^{M}}{h}-k_{i-1/2}\frac{\varphi_{i}^{M}-\varphi_{i-1}^{M}}{h}\right)$$

$$i=1,2,\dots,N-1,\qquad n=1,2,\dots,M-1\qquad (15)$$

Boundary and initial conditions have the forms (8) and (9), respectively. The unknowns f_i are defined by the approximation (13) as in (1) for $n=M$.

NUMERICAL RESULTS AND CONCLUSION

The input data for boundary and initial conditions are given as

$$u(0,t)=h_{0}(t),\quad u(1,t)=h_{1}(t),\quad u(x,0)=0,\quad (16)$$

and the coefficients are given by

$$k(x)=0.1,\qquad f(x)=\begin{cases}1, & 0\le x\le 0.5\\ 0, & 0.5\le x\le 1\end{cases}.\qquad (18)$$

By solving the direct problem with these data the solution values of $\varphi(x)$ at $t=T=1$ were recorded. Then the inverse problem was solved with this overspesi_cation to determine the unknown source $f(x)$ by the considered numerical methods. Results of recovered $f(x)$ are presented in Figs.1-2, where the symbols corresponds to approximate results. As seen from the figures, the results for the method I approximate better f(x) from (17) than the method II.

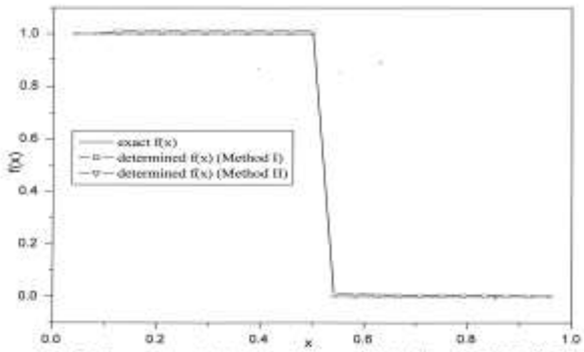

Fig.1. The exact and determined values of f(x) with grids N=25, M=25.

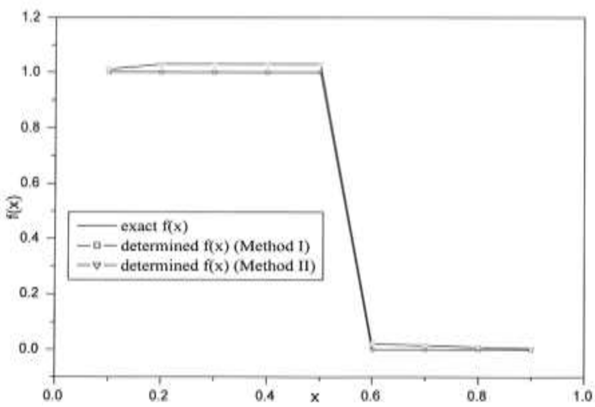

Fig.2. The exact and determined values of f(x) with grids N=11, M=11.

Next, in order to control the sensitivity of procedures to noise, artificial errors were introduced into the overspecified condition data (4) by defining the functions

$$\varphi_{\delta}(x) = \varphi(x) + 2\delta(\sigma(x) - 0.5), \quad x \in \omega,$$

where δ is the level of errors (maximum value of possible errors) and $\sigma(x)$ is a random function uniformly distributed
on [0,1]. It is observed that although the results obtained by the methods I and II are not different for small errors, method II is less sensitive to random errors than method I for large errors. In Figs.3-4, results with $\delta = 15 \times 10^{-3}$ and $\delta = 15 \times 10^{-2}$ are illustrated. The results we observed indicated that method I is a better approximate for small number of grid points than method II. On the other hand method II is less sensitive to artificial errors than method I.

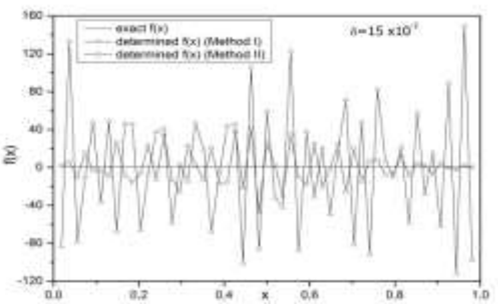

Fig.3. The exact and determined values of f(x) with M=55, N=55.

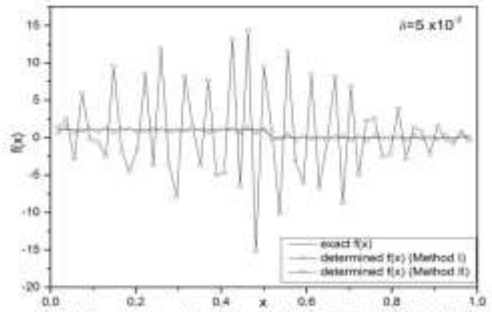

Fig.4. The exact and determined values of f(x) with M=55, N=55.

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